Journal of Applied Mechanics and Technical Physics, Vol. 49, No. 2, pp. 323–329, 2008

ANALYSIS OF THE WAVE SOLUTION OF THE ELASTOKINETIC EQUATIONS OF A COSSERAT CONTINUUM FOR THE CASE OF BULK PLANE WAVES

M. A. Kulesh, V. P. Matveenko,
M. V. Ulitin, and I. N. Shardakov

UDC 534.22.094.1

A study is made of waves in a Cosserat continuum, whose strain state is characterized by independent displacement and rotation vectors. The propagation of longitudinal and transverse bulk waves is considered. Wave solutions are sought in the form of wave trains specified by a Fourier spectrum of arbitrary shape. It is shown that if the solution is sought in the form of three components of the displacement vector and three components of the rotation vector which depend on time and the longitudinal coordinate, the initial system is split into two systems, one of which describes longitudinal waves, and the other transverse waves. For waves of both types, dispersion relations and analytical solutions in displacement are obtained. The dispersion characteristics of the solutions obtained differ from the dispersion characteristics of the corresponding classical elastic solutions.

Key words: *plane waves, dispersion, Cosserat continuum, analytical solutions.*

Introduction. Linear models of the asymmetric theory of elasticity, in particular, the Cosserat model, have been the subject of extensive studies. However, there is still no clear understanding of the role of this theory in the mechanics of deformable solids. The importance of moment theory can be determined by a correctly performed experiment using modern experimental facilities. Examples of such facilities are mechanical [1] or laser [2] sensors, which allow direct measurements of rotation velocities in three perpendicular directions. These facilities are currently employed (though not widely) in seismic and geophysical studies. Figure 1 gives an experimental six-component seismogram recorded during an underground non-nuclear explosion of an approximately 1-kton charge at a depth of 390 m and an epicenter distance of 1 km. The seismogram shows three components of the acceleration vector and independently measured components of the rotation velocity of ground at the location of the sensor [1].

In many papers, it is assumed that the components of the rotation and displacement vectors are linked by a relation that corresponds to the classical theory of elasticity or the asymmetric theory of elasticity with constrained rotation, for example, Cosserat pseudocontinuum theory (see, for example, [3]):

$$\boldsymbol{\omega} = (1/2)\,\mathrm{rot}\,\boldsymbol{u}.$$

Similar dependences with different coefficients are obtained in some dynamic problems for the reduced Cosserat continuum model [4]. In the complete linear Cosserat theory [5], the rotation $\boldsymbol{\omega}$ and displacement $\boldsymbol{u}$ vectors are kinematically independent. On the one hand, this leads to an increase in the number of necessary material parameters. On the other hand, from a physical point of view, the complete theory is more realistic than, for example, the Cosserat pseudocontinuum theory [5]. However, there are still no experimental data on the nature of the relationship between the displacement and rotation vectors, although, from the seismogram presented in Fig. 1, follows that such studies, in principle, are feasible.

Institute of Mechanics of Continuous Media, Ural Division, Russian Academy of Sciences, Perm' 614013; kma@icmm.ru; mvp@icmm.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 49, No. 2, pp. 196–203, March–April, 2008. Original article submitted March 27, 2007.

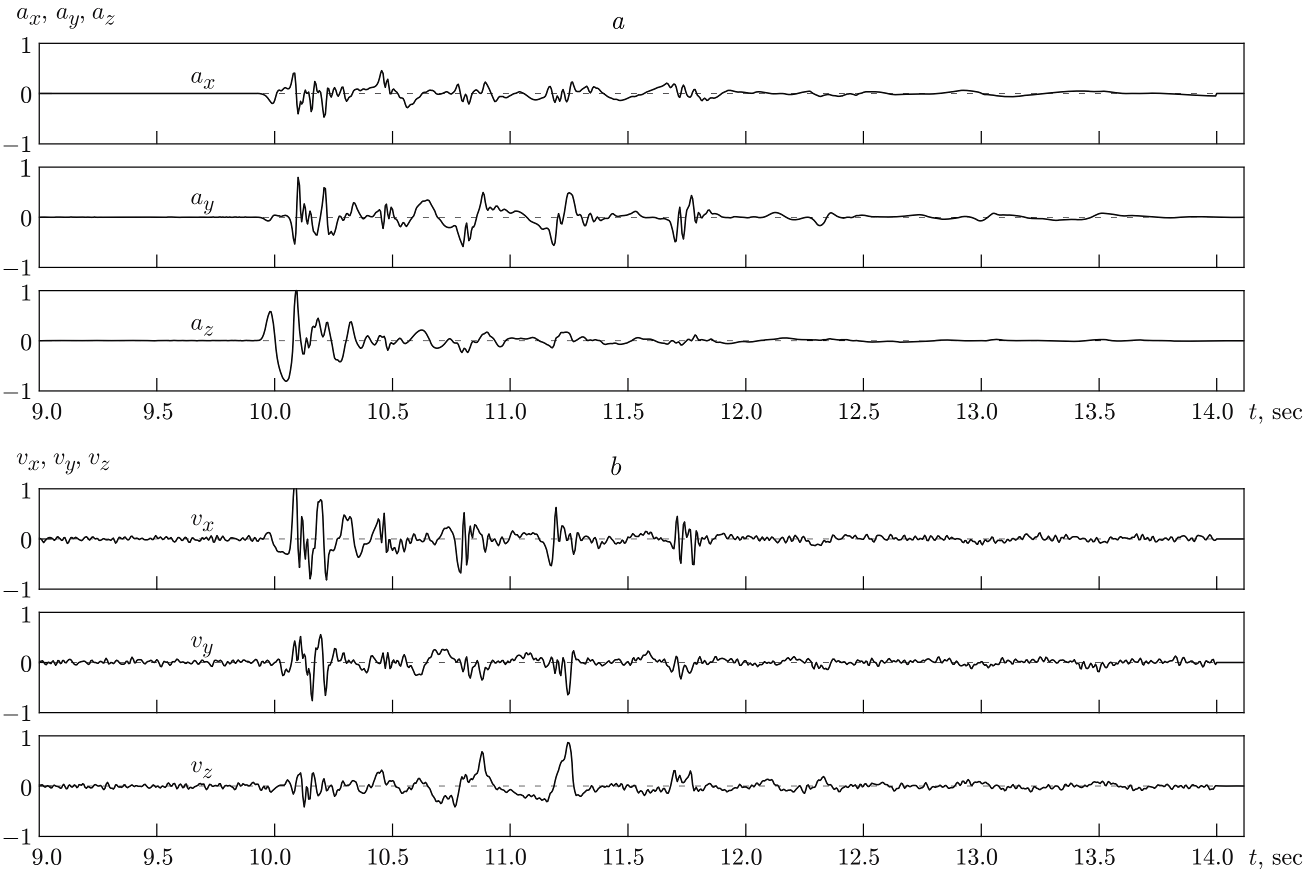

Fig. 1. Results of experimental measurements of the components of the acceleration vectors (a) and rotation velocities (b) in an underground explosion of a 1-kton charge at a depth of 390 m and an epicenter distance of 1 km [1].

Thus, wave experiments, especially in geological media, provide information for the identification of models of asymmetric media. Such experiments have been performed; in particular, results of ultrasonic studies of homogeneous media were used to identify the Le Roux model and the Cosserat pseudocontinuum model in [6] and to identify the linear Cosserat continuum model in [7, 8]. Geological media are a more complex subject of research since, in them, several types of waves, as a rule, are simultaneously excited and recorded: longitudinal and transverse direct and reflected bulk waves, Rayleigh waves, and Love waves, Lamb, and Stonely waves.

From the aforesaid, it follows that obtaining and analyzing wave solutions for various microstructural models is an urgent problem. In the present work, we continue to study the complete Cosserat model [5]. For this model, a number of new characteristic features have been determined previously. Dispersion of Rayleigh elastic surface waves was found in [9, 10] (in the classical theory of elasticity, Rayleigh waves do not exhibit dispersion). A detailed analysis of the components of the displacement and rotation vectors of Rayleigh waves was performed in [11]. An effect due to the propagation of a surface transverse wave with horizontal polarization was found. Geometrically, this wave is similar to a Love wave, but, according to the classical theory of elasticity, the existence of a Love wave as a surface wave is due to the presence of a layer on a half-space. As the thickness of the layer tends to zero, the Love wave becomes a bulk wave. It has been shown [12] that, in a Cosserat continuum, a horizontally polarized, transverse wave which decays with increasing depth exists in the absence of a plane layer. Another solution, which has no analogs in the classical theory of elasticity, was obtained in [13]. This solution describes a wave which propagates in a plate and has one transverse component of the displacement vector and two components of the rotation vector. This wave has more numbers of modes than Lamb waves, all modes possess dispersion, and the displacement in all modes depend on depth.

In the present paper, a general equation of motion for plane waves in a Cosserat continuum is considered and one more particular solution is obtained which describes the propagation of longitudinal and transverse bulk waves of displacement and rotation. Solutions of the equations of motion are obtained for the case of nonmonochromatic waves and describe the propagation of wave trains specified by a Fourier spectrum of arbitrary shape. The nonmonochromatic representation is chosen because it is the most suitable for comparison with the experimental data obtained in seismic measurements. The basic equations of a Cosserat continuum and the general equation of plane waves are also given in the paper. A particular solution for bulk waves is constructed and compared with the solutions obtained in [11–13].

1. Formulation of the Problem. We consider a space filled with an elastic isotropic medium described by the continuum Cosserat model [5]. Mass forces and moments are absent. We use Cartesian coordinates, in which a plane wave propagates in the x direction. The constitutive relations have the following form:

— the equations of motion

$$\nabla \cdot \tilde{\sigma} + \boldsymbol{X} = \rho \ddot{\boldsymbol{u}}, \qquad \tilde{\sigma}^{\mathrm{t}} : \tilde{E} + \nabla \cdot \tilde{\mu} + \boldsymbol{Y} = j\ddot{\boldsymbol{\omega}}; \tag{1.1}$$

— the geometrical relations

$$\tilde{\gamma} = \nabla \boldsymbol{u} - \tilde{E} \cdot \boldsymbol{\omega}, \qquad \tilde{\chi} = \nabla \boldsymbol{\omega}; \tag{1.2}$$

— the physical equations

$$\tilde{\sigma} = 2\mu\tilde{\gamma}^{(S)} + 2\alpha\tilde{\gamma}^{(A)} + \lambda I_1(\tilde{\gamma})\tilde{e}, \qquad \tilde{\mu} = 2\gamma\tilde{\chi}^{(S)} + 2\varepsilon\tilde{\chi}^{(A)} + \beta I_1(\tilde{\chi})\tilde{e}. \tag{1.3}$$

In view of (1.1)–(1.3), the equations of motion for the displacement vector $\boldsymbol{u}$ and the rotation vector $\boldsymbol{\omega}$ are written as

$$\begin{gathered} (2\mu + \lambda)\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} - (\mu + \alpha)\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{u} + 2\alpha\,\mathrm{rot}\,\boldsymbol{\omega} + \boldsymbol{X} = \rho\ddot{\boldsymbol{u}}, \\ (\beta + 2\gamma)\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\omega} - (\gamma + \varepsilon)\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\omega} + 2\alpha\,\mathrm{rot}\,\boldsymbol{u} - 4\alpha\boldsymbol{\omega} + \boldsymbol{Y} = j\ddot{\boldsymbol{\omega}}. \end{gathered} \tag{1.4}$$

In (1.1)–(1.4), $\boldsymbol{X}$ is the specific density vector of the bulk forces, $\boldsymbol{Y}$ is the specific density vector of the bulk moments, $\tilde{\gamma}$ and $\tilde{\chi}$ are the strain and bending-torsion tensors, $\tilde{\sigma}$ and $\tilde{\mu}$ are the stress and moment stress tensors, μ and λ are the Lamé constants, α, β, γ, and ε are physical constants of material for the elastic Cosserat model, ρ is the density, j is the density of the moment of inertia (a measure of the inertia of the medium in rotation), $\boldsymbol{E}$ is the Levy-Civita tensor of the third rank, $(\,\cdot\,)^{(S)}$ is the symmetrization operation, $(\,\cdot\,)^{(A)}$ is the alternation operation, $\nabla(\,\cdot\,)$ is the nabla-operator, $I_1(\,\cdot\,)$ is the first invariant of the tensor, and $\tilde{e}$ is the unit tensor [14]. In this model, the tensors $\tilde{\gamma}$ and $\tilde{\sigma}$ are asymmetric.

Unlike in the well-known papers [9, 10], which consider only monochromatic waves, following the procedure described, for example, in [15], we represent the general solution of system (1.4) in the form of Fourier integrals with respect to all components of the displacement vector $u_n(x, z, t)$ and the rotation vector $\omega_n(x, z, t)$:

$$u_n(x, z, t) = \int\limits_{-\infty}^{\infty} U_n(z)\,\mathrm{e}^{i(kx+ft)}\,\hat{S}_0(f)\,df, \qquad \omega_n(x, z, t) = \int\limits_{-\infty}^{\infty} W_n(z)\,\mathrm{e}^{i(kx+ft)}\,\hat{S}_0(f)\,df. \tag{1.5}$$

Here the subscript n takes values x, y, z, i is the imaginary unit, k is the wavenumber, f is the angular frequency (related to the physical frequency p, measured in Hertz, by the relation $f = 2\pi p$), t is time, $U_n(z)$ and $W_n(z)$ are amplitude functions, which depend on depth, and $\hat{S}_0(f)$ is a complex spectral function that corresponds to the Fourier spectrum of the source signal and defines the shape of the wave train. Here, only the real parts of the components of the displacement and rotation vectors have a physical meaning.

The nonmonochromatic representation (1.5) in the form of a wave train of arbitrary shape which is limited in the time and Fourier spaces is chosen to study the dispersion properties of waves and compare both the solutions and dispersion curves with experimental results similar to those presented in Fig. 1.

In this case, it is reasonable to perform a continuous Fourier transform of the equations of motion (1.4) and representation (1.5). This leads to the following system of equations for the Fourier images of the required components of the displacement and rotation vectors (it is assumed that mass forces and moments are absent):

$$\begin{gathered} (2\mu + \lambda)\mathrm{grad}\,\mathrm{div}\,\hat{\boldsymbol{u}} - (\mu + \alpha)\mathrm{rot}\,\mathrm{rot}\,\hat{\boldsymbol{u}} + 2\alpha\,\mathrm{rot}\,\hat{\boldsymbol{\omega}} + \rho f^2\hat{\boldsymbol{u}} = \boldsymbol{0}, \\ (\beta + 2\gamma)\mathrm{grad}\,\mathrm{div}\,\hat{\boldsymbol{\omega}} - (\gamma + \varepsilon)\mathrm{rot}\,\mathrm{rot}\,\hat{\boldsymbol{\omega}} + 2\alpha\,\mathrm{rot}\,\hat{\boldsymbol{u}} - (4\alpha - jf^2)\hat{\boldsymbol{\omega}} = \boldsymbol{0}. \end{gathered} \tag{1.6}$$

The Fourier transform of representation (1.5) has the form

$$\hat{\boldsymbol{u}} = \{U_x(z), U_y(z), U_z(z)\}^{\mathrm{t}}\, \mathrm{e}^{ikx}\, \hat{S}_0(f), \qquad \hat{\boldsymbol{\omega}} = \{W_x(z),\, W_y(z),\, W_z(z)\}^{\mathrm{t}}\, \mathrm{e}^{ikx}\, \hat{S}_0(f). \tag{1.7}$$

Substituting (1.7) into (1.6), we obtain two uncoupled systems of linear differential equations for the functions $U_x(z)$, $U_z(z)$, and $W_y(z)$ which describe longitudinal waves:

$$\begin{gathered}(\mu+\alpha)U_x''(z) + (\rho f^2 - k^2(\lambda+2\mu))U_x(z) + ik(\lambda+\mu-\alpha)U_z'(z) - 2\alpha W_y'(z) = 0,\\ (\lambda+2\mu)U_z''(z) + (\rho f^2 - k^2(\mu+\alpha))U_z(z) + ik(\lambda+\mu-\alpha)U_x'(z) + 2ik\alpha W_y(z) = 0,\\ (\gamma+\varepsilon)W_y''(z) + (jf^2 - k^2(\gamma+\varepsilon) - 4\alpha)W_y(z) + 2\alpha U_x'(z) - 2ik\alpha U_z(z) = 0,\end{gathered} \tag{1.8}$$

and two uncoupled systems of linear differential equations for $U_y(z)$, $W_x(z)$, and $W_z(z)$ which describe transverse waves:

$$\begin{gathered}(\gamma+\varepsilon)W_x''(z) + (jf^2 - k^2(\beta+2\gamma) - 4\alpha)W_x(z) + ik(\beta+\gamma-\varepsilon)W_z'(z) - 2\alpha U_y'(z) = 0,\\ (\beta+2\gamma)W_z''(z) + (jf^2 - k^2(\gamma+\varepsilon) - 4\alpha)W_z(z) + ik(\beta+\gamma-\varepsilon)W_x'(z) + 2ik\alpha U_y(z) = 0,\\ (\mu+\alpha)U_y''(z) + (\rho f^2 - k^2(\mu+\alpha))U_y(z) + 2\alpha W_x'(z) - 2ik\alpha W_z(z) = 0.\end{gathered} \tag{1.9}$$

Systems (1.8) and (1.9) admit solutions of three types. By choosing boundary conditions, it is possible to obtain solutions for Rayleigh waves in a half-space which decay with increasing depth [11, 12], for Lamb waves in a plate which do not decay with increasing depth [13], and for plane bulk waves with amplitude independent of depth (constant with depth). In the present paper, solutions of the third type are studied for the purpose of interpreting the quantities included in the solutions of the first and second types.

2. Construction and Analysis of the Solution. Solutions for longitudinal bulk waves are obtained from the conditions $U_x(z) = U_x$, $U_y(z) = 0$, $U_z(z) = 0$, $W_x(z) = W_x$, $W_y(z) = 0$, and $W_z(z) = 0$. Substituting this conditions into Eqs. (1.8) and (1.9), we obtain two independent dispersion equations, the first of which corresponds to longitudinal displacement waves, and the second to longitudinal rotation waves:

$$(\rho f^2 - k^2(\lambda+2\mu))U_x(z) = 0, \qquad (jf^2 - k^2(\beta+2\gamma) - 4\alpha)W_x(z) = 0.$$

These equations leads to two dispersion dependences:

$$k_1(f) = f\sqrt{\frac{\rho}{\lambda+2\mu}}, \qquad k_2(f) = \sqrt{\frac{jf^2}{\beta+2\gamma} - \frac{4\alpha}{\beta+2\gamma}}.$$

In this case, it is reasonable to convert to dimensionless variables using the dimensionless parameters C_1 and C_5:

$$\begin{gathered}k_1(f) = f/C_1, \quad k_2(f) = \sqrt{f^2/C_5^2 - k_0^2}, \quad f_1(f) = C_1 k, \quad f_2(k) = \sqrt{C_5^2 k^2 + w_0^2},\\ C_1^2 = \frac{\lambda+2\mu}{\rho X_0^2 f_0^2}, \quad C_5^2 = \frac{\beta+2\gamma}{jX_0^2 f_0^2}, \quad w_0 = 2\sqrt{\frac{\alpha}{j}}, \quad k_0 = 2\sqrt{\frac{\alpha}{\beta+2\gamma}}.\end{gathered} \tag{2.1}$$

Here X_0 is a certain characteristic dimension and f_0 is a certain characteristic frequency. Thus, in addition to the longitudinal-wave velocity C_1, it is necessary to introduce the parameter C_5 dependent on the velocity of longitudinal rotation waves. In addition, for the longitudinal rotation waves, there is a forbidden zone of frequencies characterized by the cutoff frequency w_0. From the solution considered, it also follows that the rotation waves possess dispersion, but, at high frequencies, the dispersion curve is described by the asymptotic dependence $k_r(f) = f/C_5$. The parameter C_5 was used in [11–13] to obtain solutions that correspond to Rayleigh and Lamb waves. From the results given above, its physical meaning is the limiting velocity of propagation of longitudinal rotation bulk waves.

Below, we give dependences of the wavenumber and phase velocity on frequency (2.1) for the following values of material parameters: $\lambda = 2.8\cdot 10^{10}$ N/m^2, $\mu = 4\cdot 10^9$ N/m^2, $\rho = 10^5$ kg/m^3, $\alpha = 2\cdot 10^9$ N/m^2, $\beta = 10^8$ N, $\gamma = 1.936\cdot 10^8$ N, $\varepsilon = 3.0464\cdot 10^9$ N, $j = 10^4$ kg/m, $X_0 = 1$ m, and $W_0 = 1$ rad/sec.

Dependences $k_1(f)$ and $k_2(f)$ are given in Fig. 2a. In addition to the longitudinal displacement wave with the known dispersion curve $k_1(f)$, an independent dispersing rotation wave with wavenumber $k_2(f)$ and lower frequency

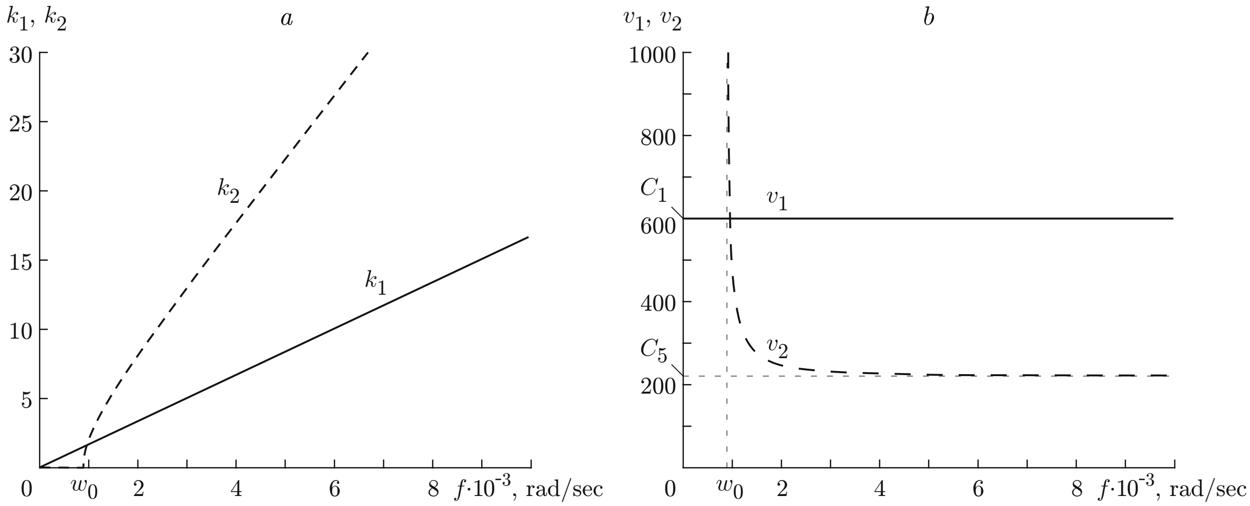

Fig. 2. Wavenumber (a) and phase velocity (b) versus frequency for longitudinal displacement and rotation waves in a Cosserat continuum.

w_0 arises in the medium. Figure 2b gives the corresponding dependences of the phase velocity on frequency. It is evident that the velocity C_5 is the limiting one for the rotation wave.

The solutions for transverse rotation bulk waves are obtained from the conditions $U_x(z) = 0$, $U_y(z) = U_y$, $U_z(z) = U_z$, $W_x(z) = 0$, $W_y(z) = W_y$, and $W_z(z) = W_z$. In this case, from Eqs. (1.8) and (1.9), we obtain two independent systems of equations for the wavenumber and frequency for horizontal and vertically polarized transverse waves, respectively:

$$\begin{aligned} &(\rho f^2 - k^2(\mu+\alpha))U_z + 2ik\alpha W_y = 0, \qquad ((\gamma+\varepsilon)k^2 + 4\alpha - jf^2)W_y + 2ik\alpha U_z = 0, \\ &(\rho f^2 - k^2(\mu+\alpha))U_y - 2ik\alpha W_z = 0, \qquad ((\gamma+\varepsilon)k^2 + 4\alpha - jf^2)W_z - 2ik\alpha U_y = 0. \end{aligned} \tag{2.2}$$

Because of isotropy of the medium, the system of equations is invariant under rotation of the coordinate system through an angle of 90°; therefore; from both systems, we obtain the same dispersion equation

$$(\gamma+\varepsilon)(\mu+\alpha)k^4 + (4\alpha\mu - (j(\mu+\alpha) + \rho(\gamma+\varepsilon))f^2)k^2 + j\rho f^4 - 4\alpha\rho f^2 = 0,$$

which can be written in dimensionless form

$$k^4 + \Big(4A^2 - \frac{C_3^2 + C_4^2}{C_3^2 C_4^2}\, f^2\Big)k^2 + \frac{f^4}{C_3^2 C_4^2} - \frac{4A^2}{C_2^2}\, f^2 = 0,$$

$$A^2 = X_0^2\, \frac{\mu\alpha}{(\mu+\alpha)(\gamma+\varepsilon)}, \qquad C_2^2 = \frac{\mu}{\rho X_0^2 f_0^2}, \qquad C_3^2 = \frac{\mu+\alpha}{\rho X_0^2 f_0^2}, \qquad C_4^2 = \frac{\gamma+\varepsilon}{j X_0^2 f_0^2}.$$

This equation has two roots:

$$k_3(f) = \sqrt{A_p}, \qquad k_4(f) = \sqrt{A_m}. \tag{2.3}$$

Here

$$A_p = \frac{C_4^2 + C_3^2}{2C_3^2 C_4^2}\, f^2 - 2A^2 + \sqrt{f^4\, \frac{C_3^4 + C_4^4 - 2C_4^2 C_3^2}{4C_3^4 C_4^4} - 2f^2\, \frac{A^2(C_3^2 C_2^2 + C_4^2 C_2^2 - 2C_4^2 C_3^2)}{C_3^2 C_4^2 C_2^2} + 4A^4},$$

$$A_m = \frac{C_4^2 + C_3^2}{2C_3^2 C_4^2}\, f^2 - 2A^2 - \sqrt{f^4\, \frac{C_3^4 + C_4^4 - 2C_4^2 C_3^2}{4C_3^4 C_4^4} - 2f^2\, \frac{A^2(C_3^2 C_2^2 + C_4^2 C_2^2 - 2C_4^2 C_3^2)}{C_3^2 C_4^2 C_2^2} + 4A^4}.$$

This solution is interpreted as follows.

1. In the case of a Cosserat continuum, a transverse wave has two wave modes with wavenumbers $k_3(f)$ and $k_4(f)$ (by virtue of isotropy of the media, the horizontally and vertically polarized transverse waves are indistinguishable; therefore, each of them has two wave modes). This feature distinguishes solutions (2.3) from the

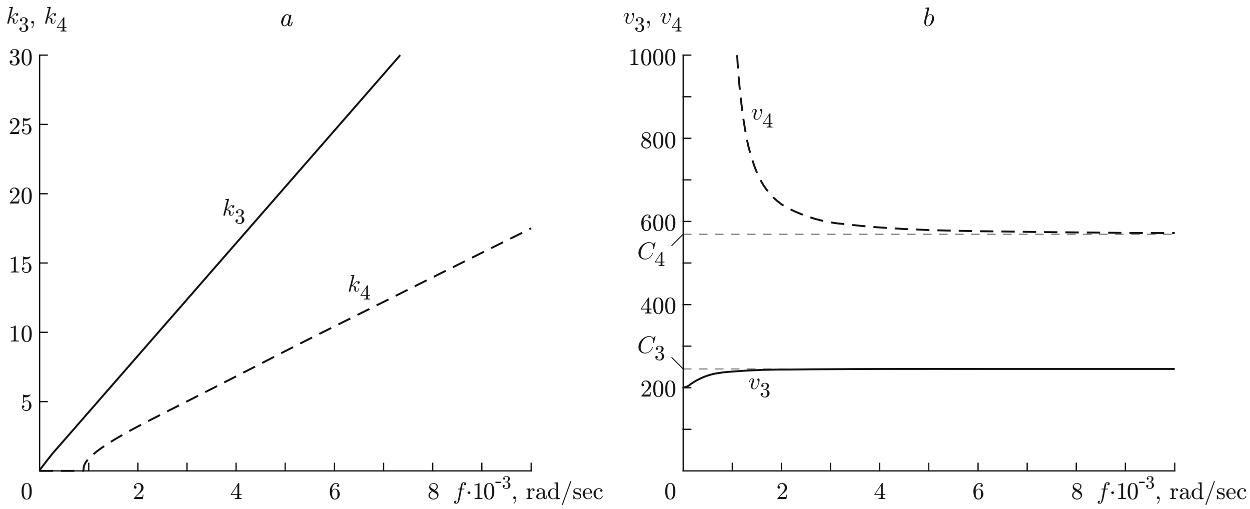

Fig. 3. Wavenumber (a) and phase velocity (b) versus frequency for transverse bulk waves in a Cosserat continuum.

classical case, where one wave mode exists. In this case, the quantities A_p and A_m in the solutions for Rayleigh waves, surface transverse waves, and Lamb waves [11, 13] are the squared wavenumbers of the two modes of the transverse bulk wave.

2. In the transverse waves, the components of the displacement and rotation vectors are related to each other by

$$W_y = iU_z \frac{\rho f^2 - k^2\mu - k^2\alpha}{2k\alpha}, \qquad W_z = iU_y \frac{-\rho f^2 + k^2\mu + k^2\alpha}{2k\alpha}$$

(the quantities U_z and U_y can take arbitrary values). These components can exist separately only for $\alpha = 0$, as follows from system (2.2).

3. Both wave modes exhibit dispersion (Fig. 3). One of the modes has the lower critical frequency [in this case, this frequency is determined from Eqs. (2.3) and is not equal to the critical frequency w_0 of the longitudinal waves]. In addition, as $f \to \infty$, the dimensionless parameters C_3 and C_4 included in solutions (2.3) are asymptotic velocities of the transverse bulk wave modes (Fig. 3b).

Conclusions. The main result obtained in this study is as follows. An interpretation was given for the parameters C_3, C_4, and C_5 included in the previously obtained solutions for surface waves [11, 13]. It was shown that, as $f \to \infty$, the dimensionless parameters C_3 and C_4 are asymptotic velocities of the transverse bulk wave modes whereas the velocity C_5 is the limiting one for the longitudinal rotation waves.

Unlike in the classical case, where transverse waves have only one wave mode, in solutions (2.1) and (2.3), four wave modes were obtained which correspond to longitudinal displacement waves, longitudinal rotation waves, and transverse waves in which the displacement directions are perpendicular to the wave propagation direction and the rotation directions are perpendicular to the wave propagation direction and the displacement direction. This result generally agrees with the results of [9]. In the case of transverse waves, the component of the vibration process corresponding to displacements exists simultaneously with the component characterized by the microrotation vector. In [9], this circumstance was ignored, which led to the erroneous conclusion that two transverse displacement and rotation waves can exist separately.

It was found that two of the four indicated wave modes have the lower critical frequency. At frequencies below this frequency, the wave cannot propagate. In the classical case and in the case of surface waves in a Cosserat continuum, this effect is not observed. However, it has been shown [4] that a similar lower critical frequency exists for Rayleigh waves within the framework of the reduced Cosserat continuum model for $\alpha = 0$. We note that experimental results in support of the presence of the forbidden frequency zones for bulk waves are not available.

This work was supported by the U.S. Civilian Research and Development Foundation (CRDF), the program of Basic Research and Higher Education (Grant No. Y2-P–09-04), and the Russian Foundation for Basic Research (Grant No. 07-01-96029-p Ural-a).

REFERENCES

1. R. L. Nigbor, "Six-degree-of-freedom ground-motion measurement," *Bull. Seismol. Soc. Amer.*, **84**, No. 5, 1665–1669 (1994).
2. H. Igel, U. Schreiber, A. Flaws, et al., "Rotational motions induced by the M8.1 Tokachi-oki earthquake, September 25, 2003," *Geophys. Res. Lett.*, **3**, L08309 (2005).
3. G. N. Savin, A. A. Lukashov, E. M. Lysko, et al., "Propagation of elastic waves in a Cosserat continuum with constrained rotation," *Prikl. Mekh.*, **6**, No. 6, 37–41 (1970).
4. M. A. Kulesh, E. F. Grekova, and I. N. Shardakov, "Rayleigh wave in the isotropic and linear, reduced Cosserat continuum," in: *Advanced Problems in Mechanics*, Book of Abstr. of Int. Summer School–Conf. (St. Petersburg, Russia, June 25–July 1, 2006), Inst. for Problems in Mech. Eng., Russian Acad. of Sci., St. Petersburg (2006), pp. 53–54.
5. W. Nowacki, *Theory of Elasticity* [Russian translation], Mir, Moscow (1975).
6. V. I. Erofeev, *Wave Processes in Solids with Microstructure* [in Russian], Izd. Mosk. Gos. Univ., Moscow (1999).
7. R. S. Lakes, "Experimental methods for study of Cosserat elastic solids and other generalized continua," in: H. Muhlhaus (ed.), *Continuum Models for Materials with Micro-Structure*, Chapter 1, J. Wiley and Sons, New York (1995), pp. 1–22.
8. R. D. Gauthier and W. E. Jahsman, "A quest for micropolar elastic constants. Part 2," *Arch. Mech.*, **33**, No. 5, 717–737 (1981).
9. A. C. Eringen, *Microcontinuum Field Theories*, Vol. 1: *Foundation and Solids*, Springer-Verlag, New York (1999).
10. A. E. Lyalin, V. A. Pirozhkov, and R. D. Stepanov, "On the propagation of surface waves in a Cosserat continuum," *Akust. Zh.*, **28**, No. 6, 838–840 (1982).
11. M. A. Kulesh, V. P. Matveenko, and I. N. Shardakov, "Construction and analysis of an analytical solution for the surface Rayleigh wave within the framework of the Cosserat continuum," *J. Appl. Mech. Tech. Phys.*, **46**, No. 4, 556–563 (2005).
12. M. A. Kulesh, V. P. Matveenko, and I. N. Shardakov, "On the propagation of elastic surface waves in a Cosserat continuum," *Akust. Zh.*, **52**, No. 2, 227–235 (2006).
13. M. A. Kulesh, V. P. Matveenko, and I. N. Shardakov, "Constructing an analytical solution for Lamb wave using the Cosserat continuum approach," *J. Appl. Mech. Tech. Phys.*, **48**, No. 1, 119–125 (2007).
14. G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers*, Dover, New York (2000).
15. P. L. Bhatnagar, *Nonlinear Waves in One-Dimensional Dispersive Systems*, Clarendon Press, Oxford (1979).